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An addition theorem for the Coulomb function

M Pauli and K Alder

Institut für Physik, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland

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Abstract. An addition theorem is derived for the regular and irregular Coulomb functions by means of the symmetry properties of the Coulomb problem in analogy to that for the spherical Bessel functions. The coefficients which enter in the addition theorem are closely related to $9j$ symbols with complex angular momenta. For the computation of these coefficients a complete set of recurrence relations is given. In addition, some useful relations are presented for the Coulomb function in configuration space as well as in momentum space.

1. Introduction

The Coulomb function and the closely related Whittaker function provide a suitable basis for the description of scattering processes in atomic and nuclear physics as well. They have been extensively discussed in the literature (Whittaker and Watson 1927, Buchholz 1953, Hull and Breit 1959, Slater 1960, Abramowitz and Stegun 1965). In this paper we present new relations for the Coulomb functions which can be useful in the theory of nuclear reactions. There, for certain purposes a good approach is to expand the Coulomb function, describing the motion of a composite particle having two components in the electrostatic field of a nucleus, in terms of products of functions which depend on the position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ of the two components. Therefore, it is our aim to derive an addition theorem where the Coulomb function depending on $\boldsymbol{\rho}_1+\boldsymbol{\rho}_2$ is expressed in terms of Coulomb functions of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$.

There exist several ways of finding an addition theorem for an arbitrary function. Obviously, one can try to write the function of $\boldsymbol{\rho}_1+\boldsymbol{\rho}_2$ as a three-dimensional Taylor series. But the disadvantage of this method lies in an asymmetric treatment of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$; one of them (e.g. $\boldsymbol{\rho}_2$) plays the role of a shift vector, while the other (e.g. $\boldsymbol{\rho}_1$) specifies the shift origin. Another operator method has been found by Sack (1964a, b, c). It allows one to write a function of the form $f(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)Y_{lm}(\widehat{\boldsymbol{\rho}_1+\boldsymbol{\rho}_2})$ in terms of functions of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$. A basically different approach is to expand the original function in terms of a complete set of easily separable functions. Since the only function F possessing the property $F(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)=F(\boldsymbol{\rho}_1)F(\boldsymbol{\rho}_2)$ is the exponential function, one must evidently write the original function as a Fourier integral. The vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are now treated symmetrically, but only at the price of leaving the original basis. On the other hand, we can expand the exponentials (i.e. the plane waves) in terms of a complete set of orthogonal functions which are better adapted to the original basis than the plane waves. Sawaguri and Tobocman (1967) have chosen the harmonic oscillator functions as the new basis. We propose to use Coulomb functions instead.

For some special functions there exists an addition theorem in the desired form, e.g. for the power function (Sack 1964) and for the spherical Bessel and Hankel functions (Danos and Maximon 1965, Buttle and Goldfarb 1966). These addition theorems can be deduced in the ways mentioned above. The relatively simple form of the addition theorem for spherical Bessel functions can be explained by means of the underlying symmetry properties of these functions. The spherical Bessel functions are solutions of a Schrödinger equation with constant potential, which is invariant with respect to translations in three-dimensional space. Therefore, the Bessel functions are closely connected to the irreducible representation of the translation group in three-dimensional space, and this gives rise to the simple structure of the addition theorem. The Coulomb problem, on the other hand, is invariant under a more complicated symmetry group, namely the rotation group in four dimensions, as Fock has shown (Fock 1935). From this invariance an addition theorem for the regular and irregular Coulomb functions in the desired form can be derived analogously to the spherical Bessel functions. In the following sections we present the relations used in the derivation of the addition theorem. In § 2 some general features of the Coulomb function in configuration space are given. In § 3 the Coulomb functions in momentum space are introduced, which are closely related to the so called hyperspherical functions. These functions are extensively discussed in § 4. Finally, the addition theorem for the regular and irregular Coulomb function is given in §§ 5 and 6.

2. The Coulomb function in configuration space

The motion of a non-relativistic particle of charge Z_1e, mass M and energy E in the Coulomb potential Z_2e/r is described by the Schrödinger equation

$$\left(-\frac{\hbar^2}{2M}\Delta+\frac{Z_1Z_2e^2}{r}-E\right)\psi(\boldsymbol{r})=0. \tag{2.1}$$

Since the Coulomb potential is spherically symmetric we can give a particular solution of this equation in the form

$$\psi(\boldsymbol{r})=\mathrm{Y}_{lm}(\theta,\phi)\frac{u_l(r)}{r}, \tag{2.2}$$

where $\mathrm{Y}_{lm}(\theta,\phi)$ is the spherical harmonic (Edmonds 1957). The radial wavefunction $u_l(r)$ satisfies the following differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}u_l(r)+\left(1-\frac{2\eta}{\rho}-\frac{l(l+1)}{\rho^2}\right)u_l(r)=0, \tag{2.3}$$

where the parameters are defined by

$$\rho=k_0r; \qquad k_0=\left(\frac{2ME}{\hbar^2}\right)^{1/2}; \qquad \eta=\frac{Z_1Z_2e^2}{\hbar^2k_0}M. \tag{2.4}$$

For $E>0$ the Coulomb parameter η is real, negative for attractive and positive for repulsive potentials. Solutions of equation (2.3) may be expressed as suitably normalized combinations of the two linearly independent Whittaker functions $\mathrm{W}_{-\mathrm{i}\eta,l+\frac{1}{2}}(-2\mathrm{i}\rho)$ and $\mathrm{W}_{\mathrm{i}\eta,l+\frac{1}{2}}(2\mathrm{i}\rho)$ (Buchholz 1953). As mentioned in the introduction, the Coulomb functions are Whittaker functions, normalized such that, for $\rho\to\infty$, the absolute value of the amplitude equals 1. Accordingly, we define Coulomb functions $H_l^{(+)}(\eta, \rho)$ and $H_l^{(-)}(\eta, \rho)$ which behave asymptotically like incoming and outgoing waves, in terms of the irregular Whittaker functions:

$$H_l^{(+)}(\eta, \rho) \sim \exp\{i[\rho - \eta \ln 2\rho - \tfrac{1}{2}\pi(l+1) + \sigma_l]\} \tag{2.5a}$$

$$= \exp[\tfrac{1}{2}\pi\eta + i\sigma_l - i\tfrac{1}{2}\pi(l+1)]W_{-i\eta, l+\frac{1}{2}}(-2i\rho) \tag{2.5b}$$

and

$$H_l^{(-)}(\eta, \rho) \sim \exp\{-i[\rho - \eta \ln 2\rho - \tfrac{1}{2}\pi(l+1) + \sigma_l]\} \tag{2.6a}$$

$$= \exp[\tfrac{1}{2}\pi\eta - i\sigma_l + i\tfrac{1}{2}\pi(l+1)]\, W_{i\eta, l+\frac{1}{2}}(2i\rho). \tag{2.6b}$$

Here, the Coulomb phase-shift σ_l is given by

$$e^{2i\sigma_l} = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)}. \tag{2.7}$$

At this point we note that the definitions (2.5) and (2.6) of $H_l^{(+)}$ and $H_l^{(-)}$ differ from the corresponding definitions of Hull and Breit (1959) by factors i and −i, respectively. According to our definitions, for $\eta = 0$ the functions are connected with the corresponding spherical Hankel functions† $h_l^{(\pm)}(\rho)$ by

$$H_l^{(\pm)}(0, \rho) = \rho h_l^{(\pm)}(\rho). \tag{2.8}$$

The functions $H_l^{(+)}$ and $H_l^{(-)}$ are linearly independent solutions of equation (2.3) satisfying the relation

$$H_l^{(+)}(\eta^*, \rho^*) = H_l^{(-)}(\eta, \rho)^*. \tag{2.9}$$

They are irregular functions with the behaviour ρ^{-l} at the origin. By a linear superposition of $H_l^{(+)}$ and $H_l^{(-)}$ we get the regular solution

$$F_l(\eta, \rho) = \tfrac{1}{2}(H_l^{(+)}(\eta, \rho) + H_l^{(-)}(\eta, \rho)). \tag{2.10}$$

This function behaves like $\sin(\rho - \eta \ln 2\rho - \frac{1}{2}\pi l + \sigma_l)$ for $\rho \to \infty$ and is connected with the regular Whittaker function $M_{i\eta, l+\frac{1}{2}}(2i\rho)$ in the following way:

$$F_l(\eta, \rho) = 2^{-(l+1)}\, e^{-i\frac{1}{2}\pi(l+1)} C_l(\eta) M_{i\eta, l+\frac{1}{2}}(2i\rho), \tag{2.11}$$

where the Coulomb penetration factor $C_l(\eta)$ is given by

$$C_l(\eta) = 2^l \frac{e^{-\frac{1}{2}\pi\eta}|\Gamma(l+1+i\eta)|}{\Gamma(2l+2)}. \tag{2.12}$$

The regular Coulomb function $F_l(\eta, \rho)$ satisfies the following orthogonality relations, derived in appendix 2:

$$\int_0^\infty d\rho\, F_l(\eta/x, x\rho) F_l(\eta/x', x'\rho) = \tfrac{1}{2}\pi\, \delta(x - x'), \tag{2.13a}$$

$$\int_0^\infty d\rho\, F_l(\eta, \rho) \frac{1}{\rho} F_l(\eta', \rho) = \tfrac{1}{2}\pi\, \delta(\eta - \eta'), \tag{2.13b}$$

† The functions $h_l^{(+)}(\rho)$ and $h_l^{(-)}(\rho)$ are denoted by $h_l^{(1)}(\rho)$ and $h_l^{(2)}(\rho)$, respectively, by Abramowitz and Stegun (1965).

$$\int_0^\infty d\rho\, F_l(\eta,\rho)\frac{1}{\rho^2}F_{l'}(\eta,\rho)=\frac{\pi}{2}\frac{1}{2l+1}\,\delta_{ll'}. \tag{2.13c}$$

There exist many recurrence relations for the Coulomb functions and the Whittaker functions, for which we refer to the literature (Buchholz 1953, Abramowitz and Stegun 1965, Slater 1960).

3. The Coulomb functions in momentum space

We get the Coulomb functions in momentum space (Guth and Mullin 1951) by Fourier analysing the solution of equation (2.1):

$$\psi(\boldsymbol{r})=\int d\boldsymbol{k}\,\psi_p(\boldsymbol{k})\,e^{i\boldsymbol{k}.\boldsymbol{r}} \tag{3.1}$$

with

$$\psi_p(\boldsymbol{k})=\frac{1}{(2\pi)^3}\lim_{\epsilon\to 0^+}\int d\boldsymbol{r}\,e^{-i\boldsymbol{k}.\boldsymbol{r}-\epsilon r}\psi(\boldsymbol{r}). \tag{3.2}$$

Here we have introduced a cut-off factor $e^{-\epsilon r}$ in order to define the integration as $r\to\infty$. The transformed Coulomb function with momentum $\boldsymbol{p}$ satisfies the following Schrödinger equation, which now is an integral equation:

$$(\boldsymbol{k}^2-k_0^2)\psi_p(\boldsymbol{k})+\frac{k_0\eta}{\pi^2}\int d\boldsymbol{k}'\frac{\psi_p(\boldsymbol{k}')}{|\boldsymbol{k}-\boldsymbol{k}'|^2}=0. \tag{3.3}$$

Expanding the solution of this equation in partial waves

$$\psi_p(\boldsymbol{k})=\sum_{lm}e^{i\sigma_l}\psi_l(\eta,k)Y^*_{lm}(\hat{k}_0)Y_{lm}(\hat{k}), \tag{3.4}$$

we obtain the regular Coulomb function $\psi_l(\eta,k)$ in momentum space (Dolinskii and Mukhamedzhanov 1965, Anni *et al* 1972) as

$$\psi_l(\eta,k)=-\frac{2}{\pi k}e^{-\frac{1}{2}\pi\eta}|\Gamma(1+i\eta)|\,\mathrm{Im}\Big[e^{-i(\sigma_l-\sigma_0)}\frac{(k+k_0+i\epsilon)^{-1+i\eta}}{(k-k_0+i\epsilon)^{1+i\eta}}$$
$$\times{}_2F_1\Big(-l,l+1,1-i\eta;-\frac{(k-k_0)^2+\epsilon^2}{4kk_0}\Big)\Big] \tag{3.5a}$$

$$\equiv-\frac{2}{\pi}\frac{\eta}{k(k-k_0-i\epsilon)(k+k_0-i\epsilon)}\exp[i\eta\arg(-2\epsilon k_0 i+k^2-k_0^2+\epsilon^2)]$$
$$\times e^{\frac{1}{2}\pi\eta}e^{-i\sigma_l}Q_l^{i\eta}\Big(\frac{k^2+k_0^2+\epsilon^2}{2kk_0}\Big). \tag{3.5b}$$

Here $Q_l^{i\eta}(Z)$ denotes the Legendre function (Abramowitz and Stegun 1965) of the second kind. Naturally, these formulae have to be taken for $\epsilon\to 0^+$. If we go to this limit, we can write

$$\psi_l(\eta,k)=-\frac{2}{\pi}\frac{\eta}{k(k^2-k_0^2)}e^{\frac{1}{2}\pi\eta-i\sigma_l}Q_l^{i\eta}\Big(\frac{k^2+k_0^2}{2kk_0}\Big)S_x(\eta) \tag{3.6}$$

where the symbol $S_x(\eta)$ means

$$S_x(\eta)=\begin{cases} e^{\pi\eta} & x<1 \\ 1 & x>1 \end{cases}, \qquad x=\frac{k}{k_0}. \tag{3.7}$$

The behaviour of the Coulomb function near the logarithmic branch point $k=k_0$ is of the following kind:

$$\psi_l(\eta,k)\sim\frac{2}{\pi}\frac{\eta}{k(k^2-k_0^2)}S_x(\eta)\,e^{-\frac{1}{2}\pi\eta}|\Gamma(1+i\eta)|\sin\left(\sigma_l-\sigma_0+\eta\ln\left|\frac{k-k_0}{k+k_0}\right|\right). \tag{3.8}$$

At the points $k=0$ and $k=\infty$ the Coulomb function $\psi_l(\eta,k)$ has the form

$$\psi_l(\eta,k)\sim\frac{2\eta}{\sqrt{\pi}}\frac{e^{-\frac{1}{2}\pi\eta}}{k_0^3}\frac{|\Gamma(l+1+i\eta)|}{\Gamma(l+\frac{3}{2})}\begin{cases} e^{\pi\eta}\left(\frac{k}{k_0}\right)^l\left[1+O\left(\frac{k}{k_0}\right)\right] & k\to 0 \\ -\left(\frac{k_0}{k}\right)^{l+4}\left[1+O\left(\frac{k_0}{k}\right)\right] & k\to\infty. \end{cases} \tag{3.9}$$

4. The spherical functions in four dimensions

As Fock has discovered in 1935, the 'accidental' degeneracy of the bound state levels of the hydrogen atom comes from a hidden symmetry of the Coulomb system, namely the symmetry with respect to the rotation group O(4) in four-dimensional space. According to Fock all the discrete states of the hydrogen atom are described by the irreducible finite-dimensional representations of the group O(4). As basis functions of these representations one can utilize the spherical functions defined on the four-dimensional sphere (Dolginov 1956, Dolginov and Toptygin 1959, Dolginov and Moskalev 1959), also called hyperspherical functions. On the other hand, the continuous states of the hydrogen atom can be given in terms of the irreducible infinite-dimensional representations of the homogeneous Lorentz group O(1, 3) (Perelomov and Popov 1966). It has been shown by Dolginov and Toptygin (1959) that an analytical continuation of the spherical functions provides the set of functions which forms a canonical basis for the infinite-dimensional representation of the Lorentz group. The properties of the hyperspherical functions have been investigated in detail by Dolginov and collaborators (1956, 1959) and by Perelomov and Popov (1966). Bander and Itzykson (1966a, b) have established the corresponding relations for the n-dimensional case. In the following we recapitulate those features of the spherical functions which we shall need for the subsequent sections.

Following Fock, the invariance of the Coulomb problem with respect to the symmetry groups mentioned above is put into evidence by embedding the three-dimensional momentum space in a four-dimensional space. This is done by a stereographic projection, by which the original momentum space is projected on a four-dimensional sphere for $E<0$, or on a two-sheeted hyperboloid for $E>0$; Let u be an arbitrary point in four-dimensional space with projection $\boldsymbol{u}$ in momentum space and fourth component u_0. The norm of u has to be invariant with respect to the symmetry group and hence it is defined as

$$u^2=u_0^2+\boldsymbol{u}^2 \qquad E<0 \tag{4.1a}$$

$$u^2=u_0^2-\boldsymbol{u}^2 \qquad E>0. \tag{4.1b}$$

If the four-dimensional sphere and the hyperboloid, respectively are given by $u^2=1$, then the point $\boldsymbol{p}$ in the original space corresponds to the point

$$u=\left(\frac{p^2\mp p_0^2}{p_0^2\pm p^2}, \frac{2p_0\boldsymbol{p}}{p_0^2\pm p^2}\right) \qquad \begin{cases}\text{upper sign} & E<0\\ \text{lower sign} & E>0\end{cases} \tag{4.2}$$

with $p_0=(2M|E|)^{1/2}$.

We shall need the following relations. If the points u and u' in the four-dimensional space correspond to $\boldsymbol{p}$ and $\boldsymbol{p}'$, respectively then

$$(\boldsymbol{p}-\boldsymbol{p}')^2=\pm\frac{(u-u')^2p_0^2}{(1\mp u_0)(1\mp u_0')}=\frac{|u-u'|^2p_0^2}{|1\mp u_0|\,|1\mp u_0'|}, \tag{4.3a}$$

$$\mathrm{d}\boldsymbol{p}=\frac{2\delta(u^2-1)\,\mathrm{d}^4u}{(1\mp u_0)^3}=\frac{p_0^3}{|1\mp u_0|^3}\,\mathrm{d}^3\mu(u) \tag{4.3b}$$

where $\mathrm{d}^3\mu(u)$ is the surface element. We define the function

$$\Phi(u)=\frac{1}{|1\mp u_0|^2}\psi(\boldsymbol{p}) \tag{4.4}$$

which satisfy the following equations (Bander and Itzykson 1966), derived from equation (3.3):

$$\Phi(u)+\frac{\eta}{2\pi^2}\int \mathrm{d}^3\mu(u')\frac{\Phi(u')}{(u-u')^2}=0 \qquad E<0 \tag{4.5a}$$

$$\Phi(u)+\epsilon(u_0)\frac{\eta}{2\pi^2}\int \mathrm{d}^3\mu(u')\frac{\Phi(u')}{|(u-u')|^2}=0 \qquad E>0 \tag{4.5b}$$

with $\epsilon(u_0)=+1$ for $u_0\geqslant 1$ and $\epsilon(u_0)=-1$ for $u_0\leqslant -1$. The invariance of equation (4.5) with respect to the groups O(4) or O(1, 3) is obvious.

Solutions of equation (4.5*b*) are the hyperspherical functions $\Psi_{\eta lm}(u)$ which are closely related to the irreducible representations of the homogeneous Lorentz group O(1, 3). Introducing in the usual way the angles on the hyperboloid:

$$u_0=\cosh\alpha; \qquad u_1=\sinh\alpha\sin\theta\cos\phi; \qquad u_2=\sinh\alpha\sin\theta\sin\phi$$
$$u_3=\sinh\alpha\cos\theta; \qquad \mathrm{d}^3\mu(u)=\sinh^2\alpha\,\mathrm{d}\alpha\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{4.6}$$

we have

$$\Psi_{\eta lm}(\alpha,\theta,\phi)=\Pi_{\eta l}(\alpha)\mathrm{Y}_{lm}(\theta,\phi). \tag{4.7}$$

The spherical harmonic $\mathrm{Y}_{lm}(\theta,\phi)$ satisfies the normalization condition

$$\sum \mathrm{Y}_{lm}(\Omega)\mathrm{Y}^*_{lm}(\Omega')=\delta(\Omega-\Omega') \tag{4.8a}$$

$$\int \mathrm{d}\Omega\,\mathrm{Y}_{lm}(\Omega)\mathrm{Y}^*_{l'm'}(\Omega)=\delta_{ll'}\,\delta_{mm'} \tag{4.8b}$$

where $\delta(\Omega-\Omega')$ is the 'δ function' on the three-dimensional sphere. The functions

$\Pi_{\eta l}(\alpha)$ are given by a hypergeometric function (Perelomov and Popov 1966)

$$\Pi_{\eta l}(\alpha)=(-1)^{l+1}\left(\frac{2}{\pi}\right)^{1/2}\frac{[\eta^2(\eta^2+1)\ldots(\eta^2+l^2)]^{1/2}}{(2l+1)!!}(\sinh\alpha)^l \times {}_2F_1\left(l+1+\mathrm{i}\eta, l+1-\mathrm{i}\eta, l+\tfrac{3}{2}; \frac{1-\cosh\alpha}{2}\right). \tag{4.9}$$

After a quadratic transformation of the ${}_2F_1$ function it is easily seen that the $\Pi_{\eta l}(\alpha)$ are related to the Legendre functions of the second kind:

$$\mathrm{e}^{\frac{1}{2}\pi\eta-\mathrm{i}\sigma_l}Q_l^{\mathrm{i}\eta}\left(\frac{p^2+p_0^2}{2pp_0}\right)=(-1)^{l+1}\left(\frac{\pi}{2}\right)^{1/2}\frac{C_0(\eta)}{\eta}\sinh\alpha\,\Pi_{\eta l}(\alpha) \tag{4.10}$$

with $p/p_0=\tanh\frac{1}{2}\alpha$ for $0\leqslant p/p_0<1$ and $p_0/p=\tanh\frac{1}{2}\alpha$ for $p/p_0>1$. By means of equations (4.10) and (3.6) the connection between the hyperspherical function $\Pi_{\eta l}(\alpha)$ and the Coulomb function in momentum space can be obtained. The following relation:

$$\Pi_{\eta l}(\alpha)=[\tfrac{1}{2}\pi\eta^2(\eta^2+1)\ldots(\eta^2+l^2)]^{-1/2}(\sinh\alpha)^l\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right)^{l+1}\cos\eta\alpha \tag{4.11}$$

can be used to get immediately the explicit expressions

$$\Pi_{\eta l}(0)=-\left(\frac{2}{\pi}\right)^{1/2}\eta\,\delta_{l0}, \tag{4.11a}$$

$$\Pi_{\eta 0}(\alpha)=-\left(\frac{2}{\pi}\right)^{1/2}\frac{\sin\eta\alpha}{\sinh\alpha}, \tag{4.11b}$$

$$\Pi_{\eta 1}(\alpha)=-\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{(\eta^2+1)^{1/2}}\frac{1}{\sinh\alpha}\left(\eta\cos\eta\alpha-\sin\eta\alpha\frac{\cosh\alpha}{\sinh\alpha}\right), \tag{4.11c}$$

$$\Pi_{\eta 2}(\alpha)=\left(\frac{2}{\pi}\right)^{1/2}\frac{1}{[(\eta^2+1)(\eta^2+4)]^{1/2}}\frac{1}{\sinh\alpha}\left[\sin\eta\alpha\left(\eta^2-\frac{2\cosh^2\alpha+1}{\sinh^2\alpha}\right)+3\eta\cos\eta\alpha\frac{\cosh\alpha}{\sinh\alpha}\right]. \tag{4.11d}$$

The $\Pi_{\eta l}(\alpha)$ have the recurrence relations:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Pi_{\eta l}(\alpha)-l\coth\alpha\,\Pi_{\eta l}(\alpha)-[\eta^2+(l+1)^2]^{1/2}\Pi_{\eta l+1}(\alpha)=0 \tag{4.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Pi_{\eta l}(\alpha)+(l+1)\coth\alpha\,\Pi_{\eta l}(\alpha)+(\eta^2+l^2)^{1/2}\Pi_{\eta l-1}(\alpha)=0 \tag{4.12b}$$

$$(\eta^2+l^2)^{1/2}\Pi_{\eta l-1}(\alpha)+(2l+1)\coth\alpha\,\Pi_{\eta l}(\alpha)+[\eta^2+(l+1)^2]^{1/2}\Pi_{\eta l+1}(\alpha)=0 \tag{4.12c}$$

and obey the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}+2\coth\alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}-\frac{l(l+1)}{\sinh^2\alpha}+\eta^2+1\right)\Pi_{\eta l}(\alpha)=0. \tag{4.13}$$

The hyperspherical functions $\Psi_{\eta lm}(\alpha, \theta, \phi)$ form a complete system on each sheet of the hyperboloid with the orthogonality relation (Bander and Itzykson 1966b):

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \sinh^2\alpha \, \mathrm{d}\alpha \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \Psi_{\eta lm}(\alpha, \theta, \phi)\Psi^*_{\eta' l' m'}(\alpha, \theta, \phi) = \delta(\eta-\eta')\,\delta_{ll'}\,\delta_{mm'} \tag{4.14}$$

and the closure relation

$$\int_0^\infty \mathrm{d}\eta \sum_{lm} \Psi_{\eta lm}(u)\Psi^*_{\eta lm}(u') = \delta_{\mathrm{hyp}}(u-u') \tag{4.15}$$

where δ_{hyp} is the 'δ function' on the hyperboloid. This closure relation is derived from the addition theorem of the hyperspherical function

$$\sum_{lm} \Psi_{\eta lm}(u)\Psi^*_{\eta lm}(u') = -\frac{\eta}{(2\pi)^{3/2}}\Pi_{\eta 0}(\alpha) \tag{4.16}$$

with $uu' = \cosh\alpha$.

The hyperspherical functions possess a multiplication theorem (Clebsch–Gordan expansion) derived by Dolginov and Toptygin (1959):

$$\begin{aligned}\Psi_{\eta_1 l_1 m_1}&(\alpha, \theta, \phi)\Psi_{\eta_2 l_2 m_2}(\alpha, \theta, \phi)\\ &= \frac{1}{(4\pi)^{1/2}}\int_0^\infty \mathrm{d}\eta \sum_{lm} \hat{l}\hat{l}_1\hat{l}_2 B(\eta, \eta_1, \eta_2)\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_1 & l_2 \\ m & m_1 & m_2\end{pmatrix}\\ &\quad\times\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2\end{bmatrix}\Psi^*_{\eta lm}(\alpha, \theta, \phi)\end{aligned} \tag{4.17}$$

where $B(\eta, \eta_1, \eta_2)$ is given by

$$\begin{aligned}B(\eta, \eta_1, \eta_2) &= -\frac{1}{(8\pi)^{1/2}}\\ &\times\frac{\sinh\pi\eta \sinh\pi\eta_1 \sinh\pi\eta_2}{\cosh\frac{1}{2}\pi(\eta+\eta_1+\eta_2)\cosh\frac{1}{2}\pi(\eta-\eta_1-\eta_2)\cosh\frac{1}{2}\pi(\eta-\eta_1+\eta_2)\cosh\frac{1}{2}\pi(\eta+\eta_1-\eta_2)}.\end{aligned} \tag{4.18}$$

The symbol $\left[\begin{smallmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2\end{smallmatrix}\right]$, hereafter denoted as the O(1, 3) coefficient, is related to the Wigner $9j$ symbol with complex angular momenta j_i by

$$\begin{Bmatrix} j & j & l \\ j_1 & j_1 & l_1 \\ j_2 & j_2 & l_2\end{Bmatrix} = \frac{1}{\surd(\mathrm{i}\eta)\surd(\mathrm{i}\eta_1)\surd(\mathrm{i}\eta_2)}\begin{pmatrix} l & l_1 & l_2 \\ 0 & 0 & 0\end{pmatrix}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2\end{bmatrix} \qquad l+l_1+l_2 \text{ even}. \tag{4.19}$$

The complex angular momenta j_i are connected with the Coulomb parameter η_i by the relation

$$2j_i + 1 = \mathrm{i}\eta_i. \tag{4.20}$$

The O(1, 3) coefficient is invariant under a permutation of its columns and it is independent of the sign of the parameters η_i. It can be factorized in the following way:

$$\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2\end{bmatrix} = \prod_{k=1}^{l}(\eta^2+k^2)^{-1/2}\prod_{m=l}^{l_1}(\eta_1^2+m^2)^{-1/2}\prod_{n=1}^{l_2}(\eta_2^2+n^2)^{-1/2}\begin{bmatrix} \widetilde{\eta \quad \eta_1 \quad \eta_2} \\ l \quad l_1 \quad l_2\end{bmatrix} \tag{4.21}$$

The reduced coefficient $\left[\begin{smallmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{smallmatrix}\right]$ considered as a function of η, η_1 and η_2 contains no singularities. It is a real function if the parameters η_i are real or purely imaginary.

In general, the O(1, 3) coefficients or the 9*j* symbols are complicated functions of the angular momenta, and their explicit computation is rather laborious. If one of the l_i equals zero, the 9*j* symbol reduces to a 6*j* symbol for which closed formulae are known (Edmonds 1957). Otherwise, it is possible to give explicit expressions for the lowest l indices only (see appendix 1). For the general case the calculation of the 9*j* symbols with complex arguments as a sum over 6*j* symbols (Dolginov and Toptygin 1959, Dolginov and Moskalev 1959) is not practical for numerical calculations. However, there exist recurrence relations for the 9*j* symbols (Arima *et al* 1954, Matsunobu and Takebe 1955), which can be extended by analytic continuation to complex angular momenta j_i. Combining the recurrence relations of the 9*j* symbols with those of the hyperspherical functions, we get two types of recurrence relations for the O(1, 3) coefficients: the 'maximal' recurrence relation ($l_2 = l + l_1$),

$$\begin{aligned}
&\left(\frac{\eta_2^2+(l_2+1)^2}{\eta_1^2+(l_1+1)^2}(\eta_1^2+l_1^2)(\eta_2^2+l_2^2)\right)^{1/2} l_1 \begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1-1 & l_2-1\end{bmatrix}\\
&\quad -\{[\eta_1^2+(l_1+1)^2][\eta_2^2+(l_2+1)^2]\}^{1/2}\left(\frac{l+(l_1+1)(2l_2+3)}{2l_1+3}\right.\\
&\quad \left.+\frac{1}{2}\frac{2l_1+1}{\eta_1^2+(l_1+1)^2}[l_1(l_1+1)+l_2(l_2+1)-l(l+1)+\eta_1^2+\eta_2^2-\eta^2+1]\right)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}\\
&\quad +\left(\frac{2l_2+3}{2}[(l_1+1)(l_1+2)+(l_2+1)(l_2+2)-l(l+1)+\eta_1^2+\eta_2^2-\eta^2+1]\right.\\
&\quad \left.+\frac{\eta_2^2+(l_2+1)^2}{2l_2+1}[l+(2l_1+1)(l_2+1)]\right)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1+1 & l_2+1\end{bmatrix}\\
&\quad -\{[\eta_1^2+(l_1+2)^2][\eta_2^2+(l_2+2)^2]\}^{1/2}(l_2+2)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1+2 & l_2+2\end{bmatrix}=0;
\end{aligned} \tag{4.22}$$

and the 'general' recurrence relation

$$\begin{aligned}
&[(\eta_1^2+l_1^2)(\eta_2^2+l_2^2)]^{1/2}(l_1+l_2-l)(l_1+l_2+l+1)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1-1 & l_2-1\end{bmatrix}\\
&-\{[\eta_1^2+(l_1+1)^2](\eta_2^2+l_2^2)\}^{1/2}(l-l_1+l_2)(l+l_1-l_2+1)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1+1 & l_2-1\end{bmatrix}\\
&-\{(\eta_1^2+l_1^2)[\eta_2^2+(l_2+1)^2]\}^{1/2}(l-l_2+l_1)(l+l_2-l_1+1)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1-1 & l_2+1\end{bmatrix}\\
&-(2l_1+1)(2l_2+1)[l_1(l_1+1)+l_2(l_2+1)-l(l+1)+\eta_1^2+\eta_2^2-\eta^2+1]\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}\\
&+\{[\eta_1^2+(l_1+1)^2][\eta_2^2+(l_2+1)^2]\}^{1/2}(l+l_1+l_2+2)(l_1+l_2-l+1)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1+1 & l_2+1\end{bmatrix}=0.
\end{aligned} \tag{4.23}$$

If one of the angular momenta l_i equals zero, we have a three-term recurrence relation, directly derived from the recurrence relation of the $6j$ symbols (Edmonds 1957):

$$[(\eta_1^2+l^2)(\eta_2^2+l^2)]^{1/2}\frac{l}{2l+1}\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & l-1 & l-1\end{bmatrix}$$

$$-\frac{1}{2}\{2l(l+1)+\eta_1^2+\eta_2^2-\eta^2+1\}\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & l & l\end{bmatrix}$$

$$+\{[\eta_1^2+(l+1)^2][\eta_2^2+(l+1)^2]\}^{1/2}\frac{l+1}{2l+1}\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & l+1 & l+1\end{bmatrix}=0. \qquad (4.24)$$

This set of recurrence relations allows for an easy computation of all O(1, 3) coefficients. First, the 'maximal' coefficients with $l_2=l+l_1$ are computed with equation (4.22), and the remaining ones are obtained by successive application of the 'general' relation (4.23). The starting values needed for the 'maximal' recurrence relation are given in appendix 1.

5. The addition theorem for the regular Coulomb function

After these introductory sections, let us now derive the addition theorem for the regular Coulomb function, multiplied by the spherical harmonic with the same index l:

$$\mathrm{f}_l(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})$$

$$\equiv\frac{\mathrm{F}_l(\eta,\rho)}{\rho}\mathrm{Y}_{lm}^*(\hat{\rho})=C_l(\eta)\rho^l\,\mathrm{e}^{-\mathrm{i}\rho}{}_1F_1(l+1-\mathrm{i}\eta,2l+2;2\mathrm{i}\rho)\mathrm{Y}_{lm}^*(\hat{\rho}), \qquad (5.1)$$

i.e. we want to represent the Coulomb function of the argument $|\boldsymbol{\rho}|=|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|$ in terms of products of Coulomb functions depending on the arguments ρ_1 and ρ_2, respectively. For this purpose we start by Fourier–Bessel analysing the given Coulomb function:

$$\mathrm{f}_l(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})=\int_0^\infty x^2\,\mathrm{d}x\,\psi_l(\eta,x)\mathrm{j}_l(x\rho)\mathrm{Y}_{lm}^*(\hat{\rho}) \qquad (5.2)$$

where the expansion function $\psi_l(\eta,x)$ is the Coulomb function in momentum space. Inserting into this relation the addition theorem for the spherical Bessel function (Danos and Maximon 1965) and using equation (3.6) for $\psi_l(\eta,x)$ we obtain:

$$\mathrm{f}_l(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})=(4\pi)^{1/2}\sum_{\substack{l_1m_1\\ l_2m_2}}\mathrm{i}^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix}l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}$$

$$\times\begin{pmatrix}l & l_1 & l_2\\ m & m_1 & m_2\end{pmatrix}\mathrm{Y}_{l_1m_1}(\hat{\rho}_1)\mathrm{Y}_{l_2m_2}(\hat{\rho}_2)F(x\rho_1,x\rho_2) \qquad (5.3)$$

with $\hat{x}=(2x+1)^{1/2}$ and

$$F(x\rho_1,x\rho_2)=-\frac{2\eta}{\pi}\int_0^\infty\mathrm{d}x\frac{x}{x^2-1}S_x(\eta)\,\mathrm{e}^{\frac{1}{2}\pi\eta-\mathrm{i}\sigma_l}Q_l^{\mathrm{i}\eta}\left(\frac{x^2+1}{2x}\right)\mathrm{j}_{l_1}(x\rho_1)\mathrm{j}_{l_2}(x\rho_2), \qquad (5.4)$$

where we must give some prescription to deal with the logarithmic branch point $x=1$. This is done below.

Now we expand the spherical Bessel functions in η space in terms of Coulomb functions. Using the result of appendix 3:

$$\mathrm{j}_l(x\rho)=\frac{1}{\pi x}\int_{-\infty}^{\infty}\mathrm{d}\eta\, S_x(\eta)\,\mathrm{e}^{\frac{1}{2}\pi\eta-\mathrm{i}\sigma_l}\mathrm{Q}_l^{\mathrm{i}\eta}\left(\frac{x^2+1}{2x}\right)\mathrm{f}_l(\eta, x\rho) \tag{5.5}$$

we can write equation (5.4) in the form

$$\begin{aligned}F(x\rho_1, x\rho_2)=&-\frac{2\eta}{\pi^3}\int_0^1\mathrm{d}x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{f}_{l_1}(\eta_1,\rho_1)\mathrm{f}_{l_2}(\eta_2,\rho_2)\exp[\tfrac{1}{2}\pi(\eta+\eta_1+\eta_2)\\&-\mathrm{i}(\sigma_l+\sigma_{l_1}+\sigma_{l_2})]\mathrm{Q}_l^{\mathrm{i}\eta}\left(\frac{x^2+1}{2x}\right)\mathrm{Q}_{l_1}^{\mathrm{i}\eta_1}\left(\frac{x^2+1}{2x}\right)\mathrm{Q}_{l_2}^{\mathrm{i}\eta_2}\left(\frac{x^2+1}{2x}\right)\frac{1}{2x}\\&\times\left[\mathrm{e}^{\pi(\eta+\eta_1+\eta_2)}\left(\frac{x^2+1}{x^2-1}-1\right)-1-\frac{x^2+1}{x^2-1}\right].\end{aligned} \tag{5.6}$$

Here the integral over x runs from 0 to 1. We have obtained these integration limits by splitting up the range of integration in equation (5.4) into the intervals $[0, 1]$ and $[1, +\infty)$ and inverting the latter by $x\to 1/x$. Next we change the integration variable x to the angle α on the hyperboloid, as introduced in § 4, using the relation

$$\begin{aligned}&\sinh\alpha=\frac{2x}{|x^2-1|}; &&\cosh\alpha=\frac{x^2+1}{|x^2-1|}\\&\mathrm{d}\alpha=\frac{2}{|x^2-1|}\mathrm{d}x; &&[0,1]\to[0,\infty).\end{aligned} \tag{5.7}$$

Then, substituting the Legendre functions by hyperspherical functions (see equation (4.10)) and the $3j$ symbols by an integral over three spherical harmonics, we can write equation (5.3) in the form

$$\begin{aligned}&\mathrm{f}_l(\eta,\rho)\mathrm{Y}^*_{lm}(\hat\rho)\\&=\left(\frac{2}{\pi}\right)^{1/2}C_0(\eta)\sum_{\substack{l_1m_1\\l_2m_2}}\mathrm{i}^{l_1+l_2-l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\,\mathrm{d}\eta_2\frac{C_0(\eta_1)C_0(\eta_2)}{\eta_1\eta_2}\\&\quad\times\mathrm{f}_{l_1}(\eta_1,\rho_1)\mathrm{Y}_{l_1m_1}(\hat\rho_1)\mathrm{f}_{l_2}(\eta_2,\rho_2)\mathrm{Y}_{l_2m_2}(\hat\rho_2)\\&\quad\times\int_0^{\infty}\sinh^2\alpha\,\mathrm{d}\alpha\int\mathrm{d}\Omega\,\Psi_{\eta lm}(\alpha,\theta,\phi)\Psi_{\eta_1l_1m_1}(\alpha,\theta,\phi)\Psi_{\eta_2l_2m_2}(\alpha,\theta,\phi)\\&\quad\times\lim_{\epsilon\to0^+}[\mathrm{e}^{-\epsilon\alpha}\cosh\alpha(1-\mathrm{e}^{\pi(\eta+\eta_1+\eta_2)})-1-\mathrm{e}^{\pi(\eta+\eta_1+\eta_2)}].\end{aligned} \tag{5.8}$$

In this relation we have interchanged the order of integrations over α, η_1 and η_2. This is possible since we have introduced the factor $\mathrm{e}^{-\epsilon\alpha}$ to ensure convergence at infinity of the integral over α. Further, by means of this factor we have subsequently defined the integrand in equation (5.4) at the singular point $x=1$.

Using the relation

$$\cosh\alpha\,\Pi_{\eta l}(\alpha)=\mathrm{Re}\begin{bmatrix}2\mathrm{i} & \eta & \eta+\mathrm{i}\\0 & l & l\end{bmatrix}\Pi_{\eta+\mathrm{i},l}(\alpha) \tag{5.9}$$

and the multiplication theorem of the hyperspherical functions (equation (4.17)) the integration over α is easily carried out and we get:

$$
\begin{aligned}
&f_l(\eta,\rho)Y^*_{lm}(\hat{\rho}) \\
&\quad=(4\pi)^{1/2}\sum_{\substack{l_1m_1\\l_2m_2}} i^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix} l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_1 & l_2\\ m & m_1 & m_2\end{pmatrix}Y_{l_1m_1}(\hat{\rho}_1)Y_{l_2m_2}(\hat{\rho}_2)\\
&\quad\times\left(\frac{\pi}{8}\right)^{1/2} e^{\pi\eta}C_0(\eta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta_1\, d\eta_2\, \tilde{f}_{l_1}(\eta_1,\rho_1)\tilde{f}_{l_2}(\eta_2,\rho_2)\\
&\quad\times\left(-B(\eta,\eta_1,\eta_2)\frac{1+e^{-\pi(\eta+\eta_1+\eta_2)}}{\sinh\pi\eta_1\sinh\pi\eta_2}\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}\right.\\
&\quad+\lim_{\epsilon\to 0^+}\mathrm{Re}\, B_\epsilon(\eta+i,\eta_1,\eta_2)\frac{1-e^{-\pi(\eta+\eta_1+\eta_2)}}{\sinh\pi\eta_1\sinh\pi\eta_2}\begin{bmatrix}2i & \eta & \eta+i\\ 0 & l & l\end{bmatrix}\\
&\quad\left.\times\begin{bmatrix}\eta+i & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}\right)
\end{aligned}\tag{5.10}
$$

with

$$\tilde{f}_{l_i}(\eta_i,\rho_i)=f_{l_i}(\eta_i,\rho_i)/C_0(\eta_i).\tag{5.11}$$

The coefficient $B(\eta,\eta_1,\eta_2)$ is defined in equation (4.18) and

$$
\begin{aligned}
&B_\epsilon(\eta+i,\eta_1,\eta_2)\\
&\quad=\frac{i}{(2\pi)^{3/2}}\{\psi(\tfrac{1}{2}[2+\epsilon-i(\eta-\eta_1+\eta_2)])-\psi(\tfrac{1}{2}[\epsilon+i(\eta-\eta_1+\eta_2)])\\
&\quad+\psi(\tfrac{1}{2}[2+\epsilon-i(\eta+\eta_1-\eta_2)])-\psi(\tfrac{1}{2}[\epsilon+i(\eta+\eta_1-\eta_2)])\\
&\quad-\psi(\tfrac{1}{2}[2+\epsilon-i(\eta+\eta_1+\eta_2)])+\psi(\tfrac{1}{2}[\epsilon+i(\eta+\eta_1+\eta_2)])\\
&\quad-\psi(\tfrac{1}{2}[2+\epsilon-i(\eta-\eta_1-\eta_2)])+\psi(\tfrac{1}{2}[\epsilon+i(\eta-\eta_1-\eta_2)])\}
\end{aligned}\tag{5.12}
$$

where $\psi(z)$ is the logarithmic derivative of the Γ function with simple poles at the points $z=-N$ $(N=0,1,2,\ldots)$. Therefore, the second term in equation (5.10) has singularities on the real η_2 axis (or on the η_1 axis), if we go to the limit $\epsilon\to 0^+$. Hence, in order to extract the 'pole terms' we shift the path of integration for η_2 by an amount of $+i$ or $-i$, respectively. Using the relation for $\epsilon=0$

$$B_0(\eta\pm i,\eta_1,\eta_2\pm i)=B(\eta,\eta_1,\eta_2)\tag{5.13}$$

we obtain the addition theorem in the following form for $\rho=|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|$:

$$
\begin{aligned}
&f_l(\eta,\rho)Y^*_{lm}(\hat{\rho})\\
&\quad=(4\pi)^{1/2}\sum_{\substack{l_1m_1\\l_2m_2}} i^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix} l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_1 & l_2\\ m & m_1 & m_2\end{pmatrix}\\
&\quad\times Y_{l_1m_1}(\hat{\rho}_1)Y_{l_2m_2}(\hat{\rho}_2)T_{\eta ll_1l_2}(\rho_1,\rho_2)
\end{aligned}\tag{5.14}
$$

where

$$T_{\eta l l_1 l_2}(\rho_1, \rho_2)$$

$$= \mathrm{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \, A(\eta, \eta_1, \eta_2) \tilde{\mathrm{f}}_{l_1}(\eta_1, \rho_1) \Big(\tilde{\mathrm{f}}_{l_2}(\eta_2, \rho_2) \begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2 \end{bmatrix}$$

$$- [1 + 4\pi \mathrm{i} x \delta(x)(1 - \mathrm{i}\tfrac{1}{2}\pi y \delta(y))] \tilde{\mathrm{f}}_{l_2}(\eta_2 + \mathrm{i}, \rho_2) \begin{bmatrix} 2\mathrm{i} & \eta & \eta+\mathrm{i} \\ 0 & l & l \end{bmatrix}$$

$$\times \begin{bmatrix} \eta+\mathrm{i} & \eta_1 & \eta_2+\mathrm{i} \\ l & l_1 & l_2 \end{bmatrix} \Big) \tag{5.15}$$

with

$$x = \eta_2 - \eta_1 + \eta + \mathrm{i}, \qquad y = \eta_2 + \eta_1 - \eta + \mathrm{i} \tag{5.15a}$$

and

$$A(\eta, \eta_1, \eta_2) = \frac{1}{4} \frac{C_0(\eta) \sinh \pi\eta \, \mathrm{e}^{\frac{1}{2}\pi(\eta - \eta_1 - \eta_2)}}{\cosh \frac{1}{2}\pi(\eta - \eta_1 - \eta_2) \cosh \frac{1}{2}\pi(\eta + \eta_1 - \eta_2) \cosh \frac{1}{2}\pi(\eta - \eta_1 + \eta_2)}. \tag{5.15b}$$

Equation (5.15) can be written more explicitly. Evaluating the 'δ functions' in (5.15) we obtain:

$$T_{\eta l l_1 l_2} = T^{(1)} + T^{(2)} + T^{(3)} \tag{5.16}$$

$$T^{(1)} = \mathrm{f}_{l_1}(\eta, \rho_1) \mathrm{f}_{l_2}(0, \rho_2) \, \mathrm{Re} \begin{bmatrix} 2\mathrm{i} & \eta & \eta+\mathrm{i} \\ 0 & l & l \end{bmatrix} \begin{bmatrix} \eta+\mathrm{i} & \eta & 0 \\ l & l_1 & l_2 \end{bmatrix} \tag{5.16a}$$

$$T^{(2)} = -2 \, e^{\pi\eta} C_0(\eta) \, \mathrm{Im} \int_{-\infty}^{\infty} \mathrm{d}\eta_1 \frac{\mathrm{e}^{-\pi\eta_1}}{\sinh \pi(\eta_1 - \eta)} \tilde{\mathrm{f}}_{l_1}(\eta_1, \rho_1) \tilde{\mathrm{f}}_{l_2}(\eta_1 - \eta, \rho_2) \begin{bmatrix} 2\mathrm{i} & \eta & \eta+\mathrm{i} \\ 0 & l & l \end{bmatrix}$$

$$\times \begin{bmatrix} \eta+\mathrm{i} & \eta_1 & \eta_1 - \eta \\ l & l_1 & l_2 \end{bmatrix} \tag{5.16b}$$

$$T^{(3)} = \mathrm{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \, A(\eta, \eta_1, \eta_2) \tilde{\mathrm{f}}_{l_1}(\eta_1, \rho_1) \Big(\tilde{\mathrm{f}}_{l_2}(\eta_2, \rho_2) \begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2 \end{bmatrix}$$

$$- \tilde{\mathrm{f}}_{l_2}(\eta_2 + \mathrm{i}, \rho_2) \begin{bmatrix} 2\mathrm{i} & \eta & \eta+\mathrm{i} \\ 0 & l & l \end{bmatrix} \begin{bmatrix} \eta+\mathrm{i} & \eta_1 & \eta_2+\mathrm{i} \\ l & l_1 & l_2 \end{bmatrix} \Big). \tag{5.16c}$$

The function $T_{\eta l l_1 l_2}(\rho_1, \rho_2)$ can also be obtained in another way if we use equation (5.9) correspondingly written for η_2. Then, performing essentially the same procedures as before, we get:

$$T^{(2)} + T^{(3)} = \mathrm{Re} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\eta_1 \, \mathrm{d}\eta_2 \, A(\eta, \eta_1, \eta_2) \begin{bmatrix} \eta & \eta_1 & \eta_2 \\ l & l_1 & l_2 \end{bmatrix} \tilde{\mathrm{f}}_{l_1}(\eta_1, \rho_1)$$

$$\times \Big(\tilde{\mathrm{f}}_{l_2}(\eta_2, \rho_2) - [1 - \mathrm{i}\pi(\eta_2 - \mathrm{i})\delta(\eta_2 - \mathrm{i})] \tilde{\mathrm{f}}_{l_2}(\eta_2 - \mathrm{i}, \rho_2) \begin{bmatrix} 2\mathrm{i} & \eta_2 & \eta_2 - \mathrm{i} \\ 0 & l_2 & l_2 \end{bmatrix} \Big) \tag{5.17}$$

and $T^{(1)}$ is still given by (5.15a). In the formulation (5.17) for $T^{(2)} + T^{(3)}$ there is only one O(1, 3) coefficient depending on l, l_1 and l_2, which enters in the double integral over η_1 and η_2. Its parameters η_i are all real and hence the coefficient itself is real.

Therefore, the computational expense for T_{η,ll_1l_2} can be reduced if we use equation (5.17) instead of equations (5.16). It may be of interest to look at equations (5.16) (or equation (5.17)) in special cases. When ρ_2 is equal to zero the terms $T^{(2)}$ and $T^{(3)}$ vanish on the right-hand side of equations (5.16). Similarly they vanish if the Coulomb parameter η equals zero, whereby equation (5.14) goes over into the addition theorem for the spherical Bessel function. Thus, for small values of ρ_2 or of η the term $T^{(1)}$ mainly will contribute to the right-hand side of equation (5.15). Otherwise, for arbitrary ρ_2 and η all terms contribute and have therefore to be computed. Thus, although the integrations over η_1 and η_2 converge rapidly, the addition theorem in this form is primarily practicable in the context of analytical investigations. For numerical applications, however, it is generally more useful to evaluate first the integrals over η_1 and η_2 by contour integration techniques. This will be done in the next section.

6. The addition theorem for the irregular Coulomb function

As is shown in appendix 4 the irregular Coulomb function

$$\mathrm{h}_l^{(+)}(\eta,\rho)\equiv \mathrm{H}_l^{(+)}(\eta,\rho)/\rho \tag{6.1}$$

can be written in terms of the regular Coulomb function $\mathrm{f}_l(\eta,\rho)$. Therefore, the results of § 5 for the regular function can be utilized to obtain an addition theorem for the irregular function. We start the derivation of the addition theorem for the irregular function by evaluating the integrals over η_1 and η_2 in equation (5.16) by means of residue calculations. As an example, the single steps performed in the calculation are demonstrated for the first term in equation (5.16*c*):

$$T_1^{(3)}=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,A(\eta,\eta_1,\eta_2)\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}$$
$$\times(\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta_1,\rho_1)+\tilde{\mathrm{h}}_{l_1}^{(-)}(\eta_1,\rho_1))\tilde{\mathrm{f}}_{l_2}(\eta_2,\rho_2). \tag{6.2}$$

In this equation we have replaced the regular function $\tilde{\mathrm{f}}_{l_1}(\eta_1,\rho_1)$ by

$$\tilde{\mathrm{f}}_{l_1}(\eta_1,\rho_1)=\tfrac{1}{2}(\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta_1,\rho_1)+\tilde{\mathrm{h}}_{l_1}^{(-)}(\eta_1,\rho_1)) \tag{6.3}$$

with

$$\tilde{\mathrm{h}}_{l_1}^{(\pm)}(\eta_1,\rho_1)=\mathrm{h}_{l_1}^{(\pm)}(\eta_1,\rho_1)/C_0(\eta_1). \tag{6.4}$$

For the evaluation of the double integral over η_1 and η_2, we have to know the asymptotic behaviour of $\tilde{\mathrm{h}}_l^{(+)}(\eta,\rho)$, $\tilde{\mathrm{h}}_l^{(-)}(\eta,\rho)$ and $\tilde{\mathrm{f}}_l(\eta,\rho)$ in the complex η plane. Using the asymptotic behaviour of the Whittaker function (Buchholz 1953) we find for the Coulomb function for $|\eta|\to\infty$:

$$\tilde{\mathrm{h}}_l^{(+)}(\eta,\rho)=\mp\frac{1}{\sqrt{\pi}}(-1)^l\,\mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{e}^{\pi\eta}\sinh\pi\eta\left(\frac{2}{\rho\eta}\right)^{3/4}\exp\{\mp\mathrm{i}[2\surd(-\mathrm{i}\eta)\surd(-2\mathrm{i}\rho)-\tfrac{1}{2}\pi(l+1)]\} \tag{6.5a}$$

$$\tilde{\mathrm{h}}_l^{(-)}(\eta,\rho)=\pm\frac{1}{\sqrt{\pi}}(-1)^l\,\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{\pi\eta}\sinh\pi\eta\left(\frac{2}{\rho\eta}\right)^{3/4}\exp\{\pm\mathrm{i}[2\surd(\mathrm{i}\eta)\surd(2\mathrm{i}\rho)-\tfrac{1}{2}\pi(l+1)]\} \tag{6.5b}$$

where the upper sign has to be taken for Re $\eta > 0$ and the lower sign for Re $\eta < 0$. From the relations (6.5) it can be seen that $\tilde{h}_l^{(+)}(\eta, \rho)$ tends to zero as $|\eta| \to \infty$ in the sector $0 > \arg \eta > -\pi$; and that $\tilde{h}_l^{(-)}(\eta, \rho)$ tends to zero as $|\eta| \to \infty$ in the sector $0 < \arg \eta < \pi$. Further, by means of equation (2.10) the asymptotic behaviour of the function $\tilde{f}_l(\eta, \rho)$ results from that for $\tilde{h}_l^{(+)}(\eta, \rho)$ and $\tilde{h}_l^{(-)}(\eta, \rho)$.

We first evaluate the integral over η_1 in $T_1^{(3)}$. From the asymptotic behaviour of the irregular functions $\tilde{h}_l^{(+)}(\eta, \rho)$ and $\tilde{h}_l^{(-)}(\eta, \rho)$ it follows that the value of the integral over η_1 is not changed if we close the path of integration with a semi-circle at infinity in the lower or upper half-plane, corresponding to the terms with $\tilde{h}_{l_1}^{(+)}(\eta_1, \rho_1)$ and $\tilde{h}_{l_1}^{(-)}(\eta_1, \rho_1)$, respectively. The countour integrals thus obtained are equal to $2\pi i$ times the sum of the residues of the poles of the function $B(\eta, \eta_1, \eta_2)$ at

$$\left.\begin{aligned} \eta_1 &= \eta - \eta_2 \pm i(2N_1+1) \\ \eta_1 &= -\eta + \eta_2 \pm i(2N_1+1) \\ \eta_1 &= \eta + \eta_2 \pm i(2N_1+1) \end{aligned}\right\} \qquad N_1 = 0, 1, 2, \ldots \tag{6.6}$$

and we get

$$\begin{aligned} T_1^{(3)} = -\frac{i}{2} C_0(\eta)\, e^{\pi\eta} \sum_{N_1=0}^{\infty} \int_{-\infty}^{\infty} & d\eta_2\, \tilde{f}_{l_2}(\eta_2, \rho_2) \\ \times \Bigg(& \frac{e^{-\pi\eta} \sinh \pi\eta}{\sinh \pi(\eta_2 - \eta) \sinh \pi\eta_2} \tilde{h}_{l_1}^{(+)}(\eta - \eta_2 - i(2N_1+1), \rho_1) \\ & \times \begin{bmatrix} \eta & \eta - \eta_2 - i(2N_1+1) & \eta_2 \\ l & l_1 & l_2 \end{bmatrix} \\ & - \frac{e^{-\pi\eta_2}}{\sinh \pi(\eta_2 - \eta)} \tilde{h}_{l_1}^{(+)}(\eta_2 - \eta - i(2N_1+1), \rho_1) \\ & \times \begin{bmatrix} \eta & \eta_2 - \eta - i(2N_1+1) & \eta_2 \\ l & l_1 & l_2 \end{bmatrix} \\ & + \frac{e^{-\pi(\eta+\eta_2)}}{\sinh \pi\eta_2} \tilde{h}_{l_1}^{(+)}(\eta + \eta_2 - i(2N_1+1), \rho_1) \\ & \times \begin{bmatrix} \eta & \eta_2 + \eta - i(2N_1+1) & \eta_2 \\ l & l_1 & l_2 \end{bmatrix} + \mathrm{cc} \Bigg), \end{aligned} \tag{6.7}$$

where cc stands for complex conjugate. Now we close the path for the integration over η_2 with a semi-circle at infinity in the lower and upper half-plane, respectively. The only poles with non-vanishing residues of the integrand in equation (6.7) lie at

$$\eta_2 = \pm i N_2 \qquad N_2 = 0, 1, 2, \ldots \tag{6.8}$$

since there exist the relations

$$\tilde{h}_{l_1}^{(+)}(\mp i(2N_1 + N_2 + 1), \rho_1) = 0 \qquad \text{for } 2N_1 + N_2 \geqslant l_1 \tag{6.9}$$

and

$$\begin{bmatrix} \eta & \mp i(2N_1 + N_2 + 1) & \eta \pm i N_2 \\ l & l_1 & l_2 \end{bmatrix} = 0 \qquad \text{for } 2N_1 + N_2 < l_1. \tag{6.10}$$

Thus the evaluation of $T_1^{(3)}$ with residue calculus, leads to

$$T_1^{(3)} = -2C_0(\eta) \sum_{N_1=0,\, N_2=0}' \left(\begin{bmatrix} \eta & \eta - \mathrm{i}(2N_1+N_2+1) & \mathrm{i}N_2 \\ l & l_1 & l_2 \end{bmatrix} \right.$$
$$\times \tilde{\mathrm{h}}_{l_1}^{(+)}(\eta - \mathrm{i}(2N_1+N_2+1), \rho_1) + \begin{bmatrix} \eta & \eta + \mathrm{i}(2N_1+N_2+1) & \mathrm{i}N_2 \\ l & l_1 & l_2 \end{bmatrix}$$
$$\left. \times \tilde{\mathrm{h}}_{l_1}^{(-)}(\eta + \mathrm{i}(2N_1+N_2+1), \rho_1) \right) \tfrac{1}{2}(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2, \rho_2) + \tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2, \rho_2)). \tag{6.11}$$

The prime at the summation sign indicates that the term with $N_2 = 0$ has to be multiplied by one half. A necessary condition for the convergence of the sum is the prescription $\rho_1 > \rho_2$. This restriction can easily be seen recalling the asymptotic behaviour of the regular and the irregular functions for large parameters η. Next, we use the expansion of the irregular Coulomb function $\mathrm{h}_l^{(+)}(\eta, \rho)$ in terms of the regular function $\mathrm{f}_l(\eta, \rho)$, given in appendix 4:

$$\mathrm{h}_l^{(+)}(\eta, \rho) = \int_0^\infty \mathrm{d}x\, a_l(\eta, x) \mathrm{f}_l(\eta/x, x\rho) \tag{6.12}$$

with

$$a_l(\eta, x) = -\mathrm{i}\frac{2}{\pi}\frac{x^{l+2}}{x^2-1} \exp\left[\frac{\pi}{2}\eta\left(1-\frac{1}{x}\right)\right] \frac{|\Gamma(l+1+\mathrm{i}\eta/x)|}{|\Gamma(l+1+\mathrm{i}\eta)|}. \tag{6.13}$$

We insert this relation into equation (6.11), where we have to substitute the Coulomb parameter η by η/x, ρ_1 by $x\rho_1$, and ρ_2 by $x\rho_2$. By means of the symmetry relations

$$a_l(\eta, -x) = (-1)^l\, \mathrm{e}^{\pi\eta/x} a_l(\eta, x) \tag{6.14}$$

and

$$\mathrm{h}_l^{(-)}(-\eta/x, -x\rho) = (-1)^l\, \mathrm{e}^{-\pi\eta/x} \mathrm{h}_l^{(+)}(\eta/x, x\rho) \tag{6.15}$$

we can extend the range of integration over x up to $-\infty$ and we get

$$\int_0^\infty \mathrm{d}x\, a_l(\eta, x) T_1^{(3)}$$
$$= -\int_{-\infty}^\infty \mathrm{d}x\, a_l(\eta, x) C_0(\eta/x) \sum_{N_1=0, N_2=0}' \begin{bmatrix} \eta/x & (\eta/x) - \mathrm{i}(2N_1+N_2+1) & \mathrm{i}N_2 \\ l & l_1 & l_2 \end{bmatrix}$$
$$\times \tilde{\mathrm{h}}_{l_1}^{(+)}((\eta/x) - \mathrm{i}(2N_1+N_2+1), x\rho_1)(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2, x\rho_2) + \tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2, x\rho_2)). \tag{6.16}$$

Again, this integral is easily evaluated by contour integration technique with the result

$$\int_0^\infty \mathrm{d}x\, a_l(\eta, x) T_1^{(3)}$$
$$= -2C_0(\eta) \sum_{N_1=0, N_2=0}' \begin{bmatrix} \eta & \eta - \mathrm{i}(2N_1+N_2+1) & \mathrm{i}N_2 \\ l & l_1 & l_2 \end{bmatrix}$$
$$\times \tilde{\mathrm{h}}_{l_1}^{(+)}(\eta - \mathrm{i}(2N_1+N_2+1), \rho_1)(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2, \rho_2) + \tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2, \rho_2)). \tag{6.17}$$

We treat the remaining terms in equation (5.16) analogously and thus find for $\rho=|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|$ and $\rho_1>\rho_2$:

$$
\begin{aligned}
&\mathrm{h}_l^{(+)}(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})\\
&\quad=(4\pi)^{1/2}C_0(\eta)\sum_{\substack{l_1m_1\\l_2m_2}}\mathrm{i}^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix}l&l_1&l_2\\0&0&0\end{pmatrix}\begin{pmatrix}l&l_1&l_2\\m&m_1&m_2\end{pmatrix}\\
&\quad\times\mathrm{Y}_{l_1m_1}(\hat{\rho}_1)\mathrm{Y}_{l_2m_2}(\hat{\rho}_2)\\
&\quad\times\Bigg(\sum_{N_1=0,N_2=0}{}^{'}\begin{bmatrix}2\mathrm{i}&\eta&\eta+\mathrm{i}\\0&l&l\end{bmatrix}\begin{bmatrix}\eta+\mathrm{i}&\eta_x-\mathrm{i}&\mathrm{i}N_2\\l&l_1&l_2\end{bmatrix}\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta_x-\mathrm{i},\rho_1)\\
&\quad\times(\mathrm{f}_{l_2}(\mathrm{i}N_2,\rho_2)+\tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2,\rho_2))\\
&\quad+\sum_{N_1=1,N_2=0}{}^{'}\begin{bmatrix}2\mathrm{i}&\eta&\eta-\mathrm{i}\\0&l&l\end{bmatrix}\begin{bmatrix}\eta-\mathrm{i}&\eta_x+\mathrm{i}&\mathrm{i}N_2\\l&l_1&l_2\end{bmatrix}\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta_x+\mathrm{i},\rho_1)\\
&\quad\times(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2,\rho_2)+\tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2,\rho_2))\\
&\quad-2\sum_{N_1=0,N_2=0}{}^{'}\begin{bmatrix}\eta&\eta_x&\mathrm{i}N_2\\l&l_1&l_2\end{bmatrix}\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta_x,\rho_1)(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2,\rho_2)+\tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2,\rho_2))\Bigg)
\end{aligned}
\tag{6.18}
$$

with $\eta_x=\eta-\mathrm{i}(2N_1+N_2+1)$. The three sums over N_1 and N_2 in equation (6.18) can be combined into one sum if we employ the following recurrence relation of the O(1, 3) coefficients:

$$
\begin{aligned}
&\begin{bmatrix}2\mathrm{i}&\eta&\eta-\mathrm{i}\\0&l&l\end{bmatrix}\begin{bmatrix}\eta-\mathrm{i}&\eta_1&\eta_2\\l&l_1&l_2\end{bmatrix}+\begin{bmatrix}2\mathrm{i}&\eta&\eta+\mathrm{i}\\0&l&l\end{bmatrix}\begin{bmatrix}\eta+\mathrm{i}&\eta_1&\eta_2\\l&l_1&l_2\end{bmatrix}\\
&\quad=\begin{bmatrix}2\mathrm{i}&\eta_1&\eta_1+\mathrm{i}\\0&l_1&l_1\end{bmatrix}\begin{bmatrix}\eta&\eta_1+\mathrm{i}&\eta_2\\l&l_1&l_2\end{bmatrix}+\begin{bmatrix}2\mathrm{i}&\eta_1&\eta_1-\mathrm{i}\\0&l_1&l_1\end{bmatrix}\begin{bmatrix}\eta&\eta_1-\mathrm{i}&\eta_2\\l&l_1&l_2\end{bmatrix}
\end{aligned}
\tag{6.19}
$$

for

$$\eta_1=\eta\pm\mathrm{i}(2N_1+N_2)\qquad\text{and}\qquad\eta_2=\pm\mathrm{i}N_2 \tag{6.19a}$$

or

$$\eta_1=\pm\mathrm{i}N_1\qquad\text{and}\qquad\eta_2=\eta\pm\mathrm{i}(2N_2+N_1). \tag{6.19b}$$

Further, we utilize the following recurrence relation for the Coulomb functions:

$$\begin{bmatrix}2\mathrm{i}&\eta-\mathrm{i}&\eta\\0&l&l\end{bmatrix}u_l(\eta-\mathrm{i},\rho)=\frac{1}{1+\mathrm{i}\eta}\left\{\rho\frac{\mathrm{d}}{\mathrm{d}\rho}-\mathrm{i}(\rho-\eta+\mathrm{i})\right\}u_l(\eta,\rho) \tag{6.20}$$

with $u_l=\tilde{\mathrm{f}}_l(\eta,\rho)$ or $\tilde{\mathrm{h}}_l^{(\pm)}(\eta,\rho)$. Then, after a short calculation, we finally find the addition theorem for the irregular Coulomb function $\mathrm{h}_l^{(+)}(\eta,\rho)$ in the compact form

$(\rho=|\boldsymbol{\rho}_1+\boldsymbol{\rho}_2|)$:

$$\mathrm{h}_l^{(+)}(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})$$

$$=(4\pi)^{1/2}\sum_{\substack{l_1m_1\\l_2m_2}}\mathrm{i}^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix} l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_1 & l_2\\ m & m_1 & m_2\end{pmatrix}\mathrm{Y}_{l_1m_1}(\hat{\rho}_1)\mathrm{Y}_{l_2m_2}(\hat{\rho})$$

$$\times 2C_0(\eta)\sum_{N_1=0,N_2=0}{}'\begin{bmatrix}\eta & \tilde{\eta} & \mathrm{i}N_2\\ l & l_1 & l_2\end{bmatrix}\frac{\rho_1}{1+\tilde{\eta}^2}\left(\frac{\mathrm{d}}{\mathrm{d}\rho_1}-\tilde{\eta}\right)\tilde{\mathrm{h}}_{l_1}^{(+)}(\tilde{\eta},\rho_1)$$

$$\times(\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2,\rho_2)+\tilde{\mathrm{f}}_{l_2}(-\mathrm{i}N_2,\rho_2)) \tag{6.21}$$

with $\tilde{\eta}=\eta-\mathrm{i}(2N_1+N_2+1)$ and the restriction $\rho_1>\rho_2$. Again, the prime at the sum sign indicates that the term with $N_2=0$ has to be multiplied by one half. Remembering relation (4.21) one notes that the O(1, 3) coefficient in (6.21) contains the factor $[1\ldots(l_2^2-N_2^2)]^{-1/2}$ which is singular for $N_2\geqslant l_2$. However, this singularity is cancelled by the corresponding factor of the function $\tilde{\mathrm{f}}_{l_2}(\pm\mathrm{i}N_2,\rho_2)$.

For some purposes it may be convenient to have the addition theorem in a form where no differential operator acts on the irregular function $\tilde{\mathrm{h}}_{l_1}^{(+)}(\eta,\rho_1)$. Thus, by restarting from equation (6.18) and again using relation (6.19) we also formulate the addition theorem in the following way:

$$\mathrm{h}_l^{(+)}(\eta,\rho)\mathrm{Y}_{lm}^*(\hat{\rho})$$

$$=(4\pi)^{1/2}\sum_{\substack{l_1m_1\\l_2m_2}}\mathrm{i}^{l_1+l_2-l}\hat{l}\hat{l}_1\hat{l}_2\begin{pmatrix} l & l_1 & l_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l_1 & l_2\\ m & m_1 & m_2\end{pmatrix}\mathrm{Y}_{l_1m_1}(\hat{\rho}_1)\mathrm{Y}_{l_2m_2}(\hat{\rho}_2)$$

$$\times\left\{\begin{bmatrix}2\mathrm{i} & \eta & \eta+\mathrm{i}\\ 0 & l & l\end{bmatrix}\begin{bmatrix}\eta+\mathrm{i} & \eta & 0\\ l & l_1 & l_2\end{bmatrix}\mathrm{h}_{l_1}^{(+)}(\eta,\rho_1)\mathrm{f}_{l_2}(0,\rho_2)\right.$$

$$+C_0(\eta)\sum_{N_1=0,N_2=0}{}'\begin{bmatrix}\eta & \tilde{\eta} & \mathrm{i}N_2\\ l & l_1 & l_2\end{bmatrix}\tilde{\mathrm{h}}_{l_1}^{(+)}(\tilde{\eta},\rho_1)$$

$$\times\left(\begin{bmatrix}2\mathrm{i} & \mathrm{i}N_2 & \mathrm{i}(N_2-1)\\ 0 & l_2 & l_2\end{bmatrix}\tilde{\mathrm{f}}_{l_2}(\mathrm{i}(N_2-1),\rho_2)\right.$$

$$\left.\left.+\begin{bmatrix}2\mathrm{i} & \mathrm{i}N_2 & \mathrm{i}(N_2+1)\\ 0 & l_2 & l_2\end{bmatrix}\tilde{\mathrm{f}}_{l_2}(\mathrm{i}(N_2+1),\rho_2)-2\tilde{\mathrm{f}}_{l_2}(\mathrm{i}N_2,\rho_2)+\mathrm{cc}\right)\right\}. \tag{6.22}$$

Of course, the above equation can also be written in a form analogous to that of equation (6.21), if relation (6.20) is applied. In this formulation as well as in that of equation (6.22) the term with $N_2=1$ requires particular attention. For this parameter value the O(1, 3) coefficients have to be computed with the help of equation (6.19), written for the special case $N_2=1$:

$$\begin{bmatrix}2\mathrm{i} & \mathrm{i} & 0\\ 0 & l_2 & l_2\end{bmatrix}\begin{bmatrix}\eta & \tilde{\eta} & \mathrm{i}\\ l & l_1 & l_2\end{bmatrix}+\begin{bmatrix}2\mathrm{i} & -\mathrm{i} & 0\\ 0 & l_2 & l_2\end{bmatrix}\begin{bmatrix}\eta & \tilde{\eta} & -\mathrm{i}\\ l & l_1 & l_2\end{bmatrix}$$

$$=\begin{bmatrix}2\mathrm{i} & \eta & \eta+\mathrm{i}\\ 0 & l & l\end{bmatrix}\begin{bmatrix}\eta+\mathrm{i} & \tilde{\eta} & 0\\ l & l_1 & l_2\end{bmatrix}+\begin{bmatrix}2\mathrm{i} & \eta & \eta-\mathrm{i}\\ 0 & l & l\end{bmatrix}\begin{bmatrix}\eta-\mathrm{i} & \tilde{\eta} & 0\\ l & l_1 & l_2\end{bmatrix}. \tag{6.23}$$

We now consider equation (6.22) for special values of the parameters. In the case $\rho=\rho_1$ (i.e. $\rho_2=0$), equation (6.22) must be an identity. Thus, since we have

$$\tilde{\mathrm{f}}_{l_2}(\eta,\rho_2)=\delta_{l_2,0} \qquad \text{for } \rho_2=0 \tag{6.24}$$

and consequently the sums over N_1 and N_2 vanish identically, this implies the relation

$$\begin{bmatrix}2\mathrm{i} & \eta & \eta+\mathrm{i}\\ 0 & l & l\end{bmatrix}\begin{bmatrix}\eta+\mathrm{i} & \eta & 0\\ l & l & 0\end{bmatrix}=1. \tag{6.25}$$

For $\eta=0$, according to equations (6.9) and (6.10), again the sums over N_1, N_2 equal zero. In this case, equation (6.22) must go over into the addition theorem for the spherical Hankel function and the remaining O(1, 3) coefficients must reduce to one:

$$\begin{bmatrix}2\mathrm{i} & 0 & \mathrm{i}\\ 0 & l & l\end{bmatrix}\begin{bmatrix}\mathrm{i} & 0 & 0\\ l & l_1 & l_2\end{bmatrix}=1. \tag{6.26}$$

Further, by means of the relation

$$\mathrm{f}_l(\eta,\rho)=\mathrm{Re}\,\mathrm{h}_l^{(+)}(\eta,\rho), \tag{6.27}$$

taking the real part of equation (6.21) or of equation (6.22), we obtain the addition theorem for the regular Coulomb function in the residue representation.

The convergence of the sums over N_1 and N_2 in equations (6.21) and (6.22) generally is quite fast for $\rho_1 \gg \rho_2$. However, there is only poor convergence if the arguments ρ_1 and ρ_2 are of the same order. In this case it is advisable to use the integral representation of the addition theorem (see § 5). The integral representation (5.14) also holds for the irregular function, if on the right-hand side of this equation the regular function depending on the larger argument is replaced by the corresponding irregular function.

Appendix 1. O(1, 3) coefficients

In this appendix we give explicit expression for $\begin{bmatrix}\eta & \eta_1 & \eta_2\\ l & l_1 & l_2\end{bmatrix}$ for some special values of the angular momenta l_i:

$$\begin{bmatrix}2\mathrm{i} & \eta & \eta+\mathrm{i}\\ 0 & l & l\end{bmatrix}=\left(\frac{(l+\mathrm{i}\eta)(l+1-\mathrm{i}\eta)}{\eta(\eta+\mathrm{i})}\right)^{1/2} \tag{A.1}$$

$$\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & 0 & 0\end{bmatrix}=1 \tag{A.2}$$

$$\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & 1 & 1\end{bmatrix}=\frac{1}{2}\frac{\eta_1^2+\eta_2^2-\eta^2+1}{[(\eta_1^2+1)(\eta_2^2+1)]^{1/2}} \tag{A.3}$$

$$\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 0 & 2 & 2\end{bmatrix}=\frac{\frac{3}{8}(\eta_1^2+\eta_2^2-\eta^2+1)(\eta_1^2+\eta_2^2-\eta^2+5)-\frac{1}{2}(\eta_1^2+1)(\eta_2^2+1)}{[(\eta_1^2+1)(\eta_1^2+4)(\eta_2^2+1)(\eta_2^2+4)]^{1/2}} \tag{A.4}$$

$$\begin{bmatrix}\eta & \eta_1 & \eta_2\\ 1 & 1 & 2\end{bmatrix}=\frac{\frac{3}{8}[(\eta_2^2+1)^2-(\eta_1^2-\eta^2)^2]+\frac{1}{4}(\eta_2^2+1)(\eta_1^2+\eta^2-\eta_2^2+1)}{[(\eta^2+1)(\eta_1^2+1)(\eta_2^2+1)(\eta_2^2+4)]^{1/2}} \tag{A.5}$$

$$\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 1 & 2 & 3 \end{bmatrix} = -\frac{5}{9}\left(\frac{(\eta_1^2+1)(\eta_2^2+4)}{(\eta_1^2+4)(\eta_2^2+9)}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 1 & 0 & 1 \end{bmatrix} + \frac{1}{9}\left(\frac{\eta_2^2+4}{\eta_2^2+9}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$+\frac{5}{6}\frac{\eta_1^2+\eta_2^2-\eta^2+7}{[(\eta_1^2+4)(\eta_2^2+9)]^{1/2}}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{(A.6)}$$

$$\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 2 & 2 \end{bmatrix} = \left(\frac{\eta_1^2+1}{\eta_1^2+4}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$+\left(\frac{\eta_2^2+1}{\eta_2^2+4}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 2 & 0 \end{bmatrix} + \frac{3}{2}\frac{\eta_1^2+\eta_2^2-\eta^2-1}{[(\eta_1^2+4)(\eta_2^2+4)]^{1/2}}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 1 & 1 \end{bmatrix} \qquad \text{(A.7)}$$

$$\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 2 & 4 \end{bmatrix} = -\frac{7}{12}\left(\frac{(\eta_1^2+1)(\eta_2^2+9)}{(\eta_1^2+4)(\eta_2^2+16)}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$+\frac{1}{6}\left(\frac{\eta_2^2+9}{\eta_2^2+16}\right)^{1/2}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$+\frac{7}{8}\frac{\eta_1^2+\eta_2^2-\eta^2+9}{[(\eta_1^2+4)(\eta_2^2+16)]^{1/2}}\begin{bmatrix} \eta & \eta_1 & \eta_2 \\ 2 & 1 & 3 \end{bmatrix}. \qquad \text{(A.8)}$$

Appendix 2. The proof of the orthogonality of the regular Coulomb function

In order to derive the orthogonality relations for the regular Coulomb function (2.13), we write down the differential equation for the two functions $F_l(\eta, kr)$ and $F_{l'}(\eta', k'r)$:

$$\frac{d^2}{dr^2}F_l(\eta, kr) + \left(k^2 - \frac{2\eta k}{r} - \frac{l(l+1)}{r^2}\right)F_l(\eta, kr) = 0 \qquad \text{(A.9)}$$

$$\frac{d^2}{dr^2}F_{l'}(\eta', k'r) + \left(k'^2 - \frac{2\eta' k'}{r} - \frac{l'(l'+1)}{r^2}\right)F_{l'}(\eta', k'r) = 0. \qquad \text{(A.10)}$$

Multiplying the two equations by $F_{l'}(\eta', k'r)$ and $F_l(\eta, kr)$ respectively, subtracting the second from the first equation and integrating the resulting equation over r in the interval $[0, \infty)$, we get

$$\int_0^\infty dr\, F_{l'}(\eta', k'r)F_l(\eta, kr)\left(k^2 - k'^2 + \frac{2(\eta' k' - \eta k)}{r} + \frac{l'(l'+1) - l(l+1)}{r^2}\right) = 0. \qquad \text{(A.11)}$$

From this relation the orthogonality of the Coulomb function is easily seen. It remains to determine the normalization factor of the 'δ function' in the orthogonality relations (2.13). For this purpose we utilize the general relation for the 'Coulomb matrix elements', given by Alder and Trautmann (1971):

$$M_{l_i,l_f}^{n-1,\kappa}=\frac{1}{k_ik_f}\int_0^\infty \mathrm{d}r\, F_{l_i}(\eta_i, k_ir)\frac{e^{-\kappa r}}{r^{n+1}}F_{l_f}(\eta_f, k_fr)$$

$$=\frac{\pi\, e^{\frac{1}{2}\pi\xi}}{\sinh \pi\xi}(2k_f)^{l_f}(2k_i)^{-l_i-1}[i(k_f-k_i)-\kappa]^{l_i-l_f+n-1}$$

$$\times\frac{(2l_i)!}{(l_i-l_f+n-1)!(2l_f+1)!}\frac{|\Gamma(l_f+1+i\eta_f|}{|\Gamma(l_i+1+i\eta_i|}$$

$$\times F_2(l_f-l_i+1-n, l_f+1+i\eta_f, -l_i-i\eta_i; 2l_f+2, -2l_i; y, x)$$

$$-\frac{\pi\, e^{-\frac{1}{2}\pi\xi}}{\sinh \pi\xi}(2k_i)^{l_i}(2k_f)^{-l_f-1}[i(k_f-k_i)-\kappa]^{l_f-l_i+n-1}$$

$$\times\frac{(2l_f)!}{(l_f-l_i+n-1)!(2l_i+1)!}\frac{|\Gamma(l_i+1+i\eta_i)|}{|\Gamma(l_f+1+i\eta_f)|}$$

$$\times F_2(l_i-l_f+1-n, l_i+1-i\eta_i, -l_f+i\eta_f; 2l_i+2, -2l_f; x, y)$$

$$-\frac{\pi}{2k_ik_f}\frac{e^{-\frac{1}{2}\pi|\xi|}}{\sinh \pi\xi}\mathrm{Re}\Big(i^{l_i-l_f-n-1}(2k_i)^{i\eta_i}(2k_f)^{-i\eta_f}(k_i-k_f-i\kappa)^{i\xi+n}\frac{\exp[i(\sigma_{l_f}-\sigma_{l_i})]}{\Gamma(n+1+i\xi)}$$

$$\times F_3(-l_i-i\eta_i, -l_f+i\eta_f, l_i+1-i\eta_i, l_f+1+i\eta_f; n+1+i\xi; 1/x, 1/y)\Big) \qquad \text{(A.12)}$$

$$x=\frac{2k_i}{k_i-k_f-i\kappa}\qquad y=-\frac{2k_f}{k_i-k_f-i\kappa}\qquad \text{and}\qquad \xi=\eta_f-\eta_i.$$

With the help of this relation, equations (2.13*a*, *b*, *c*) are easily proved:

$$\text{I}\qquad \int_0^\infty \mathrm{d}r\, F_l(\eta/k, kr)F_l(\eta/k', k'r)=kk'M_{l,l}^{0,0}. \qquad \text{(A.13)}$$

From equation (A.12) we get:

$$M_{l,l}^{0,0}=-\frac{\pi}{2kk'}\lim_{\kappa\to 0^+}\frac{\xi\, e^{-\frac{1}{2}\pi|\xi|}}{\sinh \pi\xi}\mathrm{Re}\Big(i\frac{(2k)^{i\eta/k}}{(2k')^{i\eta/k'}}\frac{(k-k'-i\kappa)^{i\xi}}{k-k'-i\kappa}\frac{\exp[i(\sigma_l(\eta/k')-\sigma_l(\eta/k))]}{\Gamma(1+i\xi)}$$

$$\times F_3(-l-i\eta/k, -l+i\eta/k', l+1-i\eta/k, l+1+i\eta/k'; i\xi; 1/x, 1/y)\Big). \qquad \text{(A.14)}$$

As we have mentioned, it remains to compute the normalization factor of the orthogonality relation (2.13*a*). Hence, we take $k=k'$ in all non-singular terms of equation (A.14) and get

$$M_{l,l}^{0,0}=\frac{\pi}{2k^2}\lim_{\kappa\to 0^+}\frac{1}{\pi}\frac{\kappa}{(k-k')^2+\kappa^2}=\frac{\pi}{2k^2}\delta(k-k') \qquad \text{(A.15)}$$

which is the desired result.

$$\text{II} \qquad \int_0^\infty \mathrm{d}r\, \mathrm{F}_l(\eta, kr)\frac{1}{r}\mathrm{F}_l(\eta', kr) = k^2 M_{l,l}^{-1,0}. \tag{A.16}$$

From equation (A.12) we obtain

$$M_{l,l}^{-1,\kappa} = \frac{\pi}{2k^2} \lim_{\kappa\to 0^+} \frac{1}{\sinh \pi\xi} \mathrm{Re}\left\{\mathrm{i}(2k)^{-\mathrm{i}\xi}\kappa^{\mathrm{i}\xi}\frac{\mathrm{e}^{\mathrm{i}(\sigma_l(\eta')-\sigma_l(\eta))}}{\Gamma(1+\mathrm{i}\xi)}\right.$$
$$\times \mathrm{F}_3(-l-\mathrm{i}\eta, -l+\mathrm{i}\eta', l+1-\mathrm{i}\eta, l+1+\mathrm{i}\eta'; 1+\mathrm{i}\xi; 1/x, 1/y). \tag{A.17}$$

Now we let ξ and κ tend to zero in all non-singular terms and find

$$M_{l,l}^{-1,\kappa} = -\frac{\pi}{2k^2} \lim_{\kappa\to 0^+} \frac{\sin \xi \ln \kappa}{\sinh \pi\xi} = \frac{\pi}{2k^2}\delta(\xi). \tag{A.18}$$

$$\text{III} \qquad \int \mathrm{d}r\, \mathrm{F}_l(\eta, kr)\frac{1}{r^2}\mathrm{F}_{l'}(\eta, kr) = k^2 M_{l,l'}^{-2,0}. \tag{A.19}$$

In accordance with equation (A.11) it suffices to calculate this integral for $l = l'$:

$$M_{l,l}^{-2,0} = \frac{\pi}{2}\frac{1}{2l+1}\frac{1}{k}\frac{1}{\sinh \pi\xi}(\mathrm{e}^{\frac{1}{2}\pi\xi} - \mathrm{e}^{-\frac{1}{2}\pi\xi}) = \frac{\pi}{2}\frac{1}{2l+1}\frac{1}{k}. \tag{A.20}$$

Appendix 3. Expansion of the spherical Bessel function in terms of Coulomb functions

For the expansion of the spherical Bessel function in terms of Coulomb functions we make the following *ansatz*:

$$\mathrm{j}_l(x\rho) = \int_{-\infty}^{\infty} \mathrm{d}\eta\, \mathrm{g}_l(\eta, x)\mathrm{f}_l(\eta, \rho). \tag{A.21}$$

Using the orthogonality relation (2.13*b*) of the Coulomb functions and the relation for the 'Coulomb monopole matrix element', given by Trautmann and Alder (1970) we get:

$$\mathrm{g}_l(\eta, x) = \frac{2}{\pi}\int_0^\infty \mathrm{d}\rho\, \rho\, \mathrm{j}_l(x\rho)\mathrm{f}_l(\eta, \rho)$$
$$= \lim_{\epsilon\to 0^+} \frac{2}{\pi}\frac{\Gamma(l+1)|\Gamma(l+1+\mathrm{i}\eta)|}{\Gamma(2l+2)}\mathrm{e}^{-\frac{1}{2}\pi\eta}\left(\frac{x-1-\mathrm{i}\epsilon}{x+1-\mathrm{i}\epsilon}\right)^{\mathrm{i}\eta}$$
$$\times \frac{(-x_0)^l}{(x-1)^2+\epsilon^2}\,{}_2F_1(l+1, l+1-\mathrm{i}\eta, 2l+2; x_0) \tag{A.22}$$

with

$$x_0 = -\frac{4x}{(x-1)^2+\epsilon^2}.$$

The expansion function $\mathrm{g}_l(\eta, x)$ can also be written as a Legendre function of the second kind or a hyperspherical function

$$\mathrm{g}_l(\eta, x) = \frac{1}{\pi x} S_x(\eta)\, \mathrm{e}^{\frac{1}{2}\pi\eta}\, \mathrm{e}^{-\mathrm{i}\sigma_l} \mathrm{Q}_l^{\mathrm{i}\eta}\left(\frac{x^2+1}{2x}\right) \tag{A.23a}$$

$$=\frac{(-1)^{l+1}}{(2\pi)^{1/2}}\frac{1}{\eta x}S_x(\eta)C_0(\eta)\sinh\alpha\Pi_{\eta l}(\alpha), \tag{A.23b}$$

with $x=\tanh\frac{1}{2}\alpha$ for $|x|<1$ and $1/x=\tanh\frac{1}{2}\alpha$ for $x>1$. The function $S_x(\eta)$ is defined in equation (3.7).

We want now to evaluate the integral over η in equation (A.21), which could be useful in another context. Defining the function

$$\hat{\mathrm{f}}_l(\eta,\rho)=\mathrm{f}_l(\eta,\rho)/(\mathrm{e}^{-\frac{1}{2}\pi\eta}|\Gamma(l+1+\mathrm{i}\eta)|) \tag{A.24}$$

we can write equation (A.21) with the help of equation (4.11) in the form

$$\mathrm{j}_l(x\rho)=(-1)^l\frac{2}{x}(\sinh\alpha)^{l+1}\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right)^l\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{S_x(\eta)}{\mathrm{e}^{2\pi\eta}-1}\frac{\sin\eta\alpha}{\sinh\alpha}\hat{\mathrm{f}}_l(\eta,\rho). \tag{A.25}$$

The integrand has simple poles at the point $\eta=\pm\mathrm{i}N$. By means of contour integration equation (A.25) is easily evaluated and we get

$$\mathrm{j}_l(x\rho)=(-1)^l\frac{2}{x}(\sinh\alpha)^{l+1}\sum_{N=0}^{\infty}{}'\,S_x(\mathrm{i}N)\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right)^l\frac{\mathrm{e}^{-N\alpha}}{\sinh\alpha}\,\mathrm{Re}\,\hat{\mathrm{f}}_l(\mathrm{i}N,\rho) \tag{A.26}$$

where the prime at the summation sign indicates that the term with $N=0$ has to be multiplied by one half. We define a new set of functions

$$\mathbb{P}_{N,l}(\alpha)=\left(\frac{2}{\pi}\right)^{1/2}\sinh^l\alpha\left(\frac{\mathrm{d}}{\mathrm{d}\cosh\alpha}\right)^l\frac{\mathrm{e}^{-N\alpha}}{\sinh\alpha}. \tag{A.27}$$

The first members are

$$\mathbb{P}_{N,0}(\alpha)=\left(\frac{2}{\pi}\right)^{1/2}\frac{\mathrm{e}^{-N\alpha}}{\sinh\alpha} \tag{A.28a}$$

$$\mathbb{P}_{N,1}(\alpha)=-\left(\frac{2}{\pi}\right)^{1/2}\frac{\mathrm{e}^{-N\alpha}}{\sinh\alpha}\left(N+\frac{\cosh\alpha}{\sinh\alpha}\right) \tag{A.28b}$$

$$\mathbb{P}_{N,2}(\alpha)=\left(\frac{2}{\pi}\right)^{1/2}\frac{\mathrm{e}^{-N\alpha}}{\sinh\alpha}\left(N^2+\frac{2\cosh^2\alpha+1}{\sinh^2\alpha}+3N\frac{\cosh\alpha}{\sinh\alpha}\right). \tag{A.28c}$$

The $\mathbb{P}_{N,l}(\alpha)$ satisfy the recurrence relation

$$\mathbb{P}_{N,l+1}(\alpha)+(2l+1)\coth\alpha\,\mathbb{P}_{N,l}(\alpha)-(N^2-l^2)\,\mathbb{P}_{N,l-1}(\alpha). \tag{A.29}$$

Accordingly we can write

$$\mathrm{j}_l(x\rho)=(-1)^l(2\pi)^{1/2}\frac{1}{x}\sinh\alpha\sum_{N=0}^{\infty}{}'\,S_x(\mathrm{i}N)\mathbb{P}_{N,l}(\alpha)\,\mathrm{Re}\,\hat{\mathrm{f}}_l(\mathrm{i}N,\rho). \tag{A.30}$$

This equation can be interpreted as a 'multiplication theorem' for the spherical Bessel functions.

Appendix 4. Expansion of the irregular Coulomb function in terms of regular Coulomb functions

For $h_l^{(+)}(\eta, \rho)$ we make the following *ansatz*:

$$h_l^{(+)}(\eta, \rho) = \int_0^\infty dx\, a_l(\eta, x) f_l(\eta/x, x\rho). \tag{A.31}$$

Using the orthogonality relation (2.13*a*) of the regular Coulomb function we get

$$a_l(\eta, x) = \frac{2}{\pi} x^2 \int_0^\infty \rho^2\, d\rho\, h_l^{(+)}(\eta, \rho) f_l(\eta/x, x\rho) \tag{A.32a}$$

$$= \left(-\frac{1}{2}\right)^{l+1} e^{\frac{1}{2}\pi\eta + i\sigma_l} C_l\left(\frac{\eta}{x}\right) \frac{2x}{\pi} \int_0^\infty d\rho\, M_{i\eta/x, l+\frac{1}{2}}(2ix\rho) W_{-i\eta, l+\frac{1}{2}}(-2i\rho), \tag{A.32b}$$

where we have written the Coulomb functions in the form of Whittaker functions. The integral in equation (A.32*b*) can be solved by means of the general integral formula given by Buchholz (1953):

$$\int dz \left(\frac{a_2^2 - a_1^2}{4} + \frac{\kappa a_1 - \lambda a_2}{z} + \frac{\nu^2 - \mu^2}{z^2}\right) P_{\kappa\mu}^{(1)}(a_1 z) P_{\lambda\nu}^{(2)}(a_2 z) = \begin{vmatrix} P_{\kappa\mu}^{(1)}(a_1 z) & P_{\lambda\nu}^{(2)}(a_2 z) \\ \frac{d}{dz} P_{\kappa\mu}^{(1)}(a_1 z) & \frac{d}{dz} P_{\lambda\nu}^{(2)}(a_2 z) \end{vmatrix} \tag{A.33}$$

where $P_{ab}(z)$ is any kind of Whittaker function. In our case, we get

$$a_l(\eta, x) = \frac{2}{\pi}\left(-\frac{1}{2}\right)^{l+1} \frac{x}{x^2 - 1} e^{\frac{1}{2}\pi\eta + i\sigma_l} C_l\left(\frac{\eta}{x}\right) \left.\begin{vmatrix} M_{i\eta/x, l+\frac{1}{2}}(2ix\rho) & W_{-i\eta, l+\frac{1}{2}}(-2i\rho) \\ \frac{d}{d\rho} M_{i\eta/x, l+\frac{1}{2}}(2ix\rho) & \frac{d}{d\rho} W_{-i\eta, l+\frac{1}{2}}(-2i\rho) \end{vmatrix}\right|_0^\infty \tag{A.34}$$

The value of the determinant vanishes at the upper limit, and thus we get a contribution from the lower limit only. This leads to

$$a_l(\eta, x) = \frac{2}{i\pi} \frac{x^{l+2}}{x^2 - 1} \exp\left[\tfrac{1}{2}\pi\eta\left(1 - \frac{1}{x}\right)\right] \frac{|\Gamma(l + 1 + i\eta/x)|}{|\Gamma(l + 1 + i\eta)|}. \tag{A.35}$$

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